AIP | Journal of Mathematical Physics

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Citation: J. Math. Phys. **8**, 782 (1967); doi: 10.1063/1.1705276
View online: http://dx.doi.org/10.1063/1.1705276
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JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 4 APRIL 1967

Topology in General Relativity

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(Received 19 July 1966)

A number of theorems and definitions which are useful in the global analysis of relativistic world models are presented. It is shown in particular that, under certain conditions, changes in the topology of spacelike sections can occur if and only if the model is acausal. Two new covering manifolds, embodying certain properties of the universal covering manifold, are defined, and their application to general relativity is discussed.

INTRODUCTION

A NUMBER of theorems are proved which are useful in the analysis of general relativistic world models, particularly with regard to the problem of singularities. These are called "topological" results because they do not involve the Einstein equations directly, but rather deal with the global properties of a Lorentz signature metric on a 4-manifold. We state here the various theorems and definitions in detail, but content ourselves with a survey of possible applications.

In Sec. I, we discuss some implications of the requirement that causality be maintained, i.e., that a continuous choice of the forward light cone can be made and that no closed timelike curves exist. In Sec. II, we define two new types of covering manifolds, these adapted to the Lorentz signature metric. The applications of these and of the conventional universal covering manifold to general relativity are discussed briefly.

It is convenient to define a geometry as a 4-manifold carrying a metric of signature $(-, +, +, +)$. Modifiers will then refer to the appropriate structure. (For example, a compact geometry means the manifold is compact.) Following conventional practice, a manifold is always assumed without boundary[1] unless otherwise explicitly stated.

I. CAUSALITY

We observe[2] in our local region of space–time that from any one event P we are able to influence only those events which lie within a single cone from P, the forward light cone. Suppose we assume that this represents a universal property of space–time. Then this assumption places a global restriction on those geometries which are of physical interest. This requirement of causality may be stated mathematically: (a) no closed timelike or null curves exist, and (b) a continuous choice of the forward light cone can be made. [Following the terminology of Calabi and Markus,[3] a geometry satisfying property (b) is called isochronous.] Here, we relate these two mathematical restrictions to other, more useful, properties of the geometry.

We first require a result due to Misner.[4]

Theorem 1: Let S and S' be two compact 3-manifolds. Then there exists a compact geometry M whose boundary is the disjoint union of S and S', and in which S and S' are both spacelike.

Proof: That there exists, for any compact S and S', a compact 4-manifold M whose boundary is the disjoint union of S and S' was first proved by Rohklin[5] using cobordism theory.[6] The problem of placing a Lorentz signature metric on M is equivalent[7] to that of placing a continuous vector field on M, nowhere zero and nowhere tangent to S or S'. The Poincaré-Hopf theorem[8] states that a necessary and sufficient condition that such a vector field can be found is that the Euler characteristic,[9] χ, of M vanish.

The construction reduces, therefore, to suitably modifying M so that its Euler characteristic vanishes. Let V be some other compact 4-manifold (without boundary). Suppose we remove a small 4-ball from

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[*] National Science Foundation Predoctoral Fellow (1966).

[1] See, for example, J. R. Munkres, *Elementary Differential Topology* (Princeton University Press, Princeton, New Jersey, 1963).

[2] See, for example, H. Reichenbach, *The Direction of Time* (University of California Press, Berkeley, California, 1956); H. Reichenbach, *The Philosophy of Space and Time* (Dover Publications, Inc., New York, 1958), Sec. 21.

[3] E. Calabi and L. Markus, Ann. Math. **75**, 63 (1962).

[4] The statement and an outline of the proof of this theorem were discovered by C. W. Misner. An important step in the argument was provided by L. Markus. *Added in proof:* Essentially the same result has been obtained by B. L. Reinhart, Topology, **2**, 173 (1963).

[5] V. A. Rohklin, Dokl. Akad. Nauk USSR **81**, 355 (1951).

[6] See, for example, C. T. C. Wall, Ann. Math. **72**, 292 (1960); R. Thom, Commun. Math. Helv. **28**, 17 (1954).

[7] See, for example, N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton, New Jersey, 1951), p. 207.

[8] H. Hopf, Math. Ann. **96**, 225 (1926). See also J. Milnor, *Topology from the Differential Viewpoint* (University Press of Virginia, Charlottesville, Virginia, 1965), p. 35.

[9] See P. J. Hilton and S. Wylie, *Homology Theory* (Cambridge University Press, Cambridge, England, 1960), p. 167.

V and from the interior of M, and identify the boundaries (3-spheres) thus created. The result is a new compact 4-manifold, M', whose boundary is the same as that of M, but whose Euler characteristic differs from that of M by[10]

$$\chi(M') - \chi(M) = \chi(V) - 2.$$

Now, the 4-torus,[11] $S^1 \times S^1 \times S^1 \times S^1$, has an Euler characteristic of zero. Complex projective 2-space, a real 4-manifold, has an Euler characteristic of three. Thus, if V is chosen to be the 4-torus in the above construction, we decrease the Euler characteristic of M by two, and if V is complex projective 2-space, we increase χ by one. Therefore, a sequence of such operations can be found which reduces the Euler characteristic of M to zero.

Theorem 1 states that, physically, whatever the topology of a compact spacelike section of our space-time (assuming one exists) at the present epoch, the mere presence of a Lorentz signature metric does not preclude any given compact spacelike section's appearing at a later epoch. Theorem 2 below shows that, if such "changes in the topology" are to occur, then causality cannot be maintained.

Theorem 2: Let M be a compact geometry whose boundary is the disjoint union of two compact spacelike 3-manifolds, S and S'. Suppose M is isochronous, and has no closed timelike curve. Then S and S' are diffeomorphic, and further M is topologically $S \times [0, 1]$.

Proof: Since M is isochronous, we may construct[12] on M a timelike vector field, ξ^μ, which is nowhere zero and nowhere tangent to S or S'. Let γ be a curve in M, beginning on S, and everywhere tangent to ξ^μ. Suppose first that γ has no future endpoint. Then γ may be parametrized by a continuous variable t with range zero to infinity, $\gamma(t)$. Consider the sequence of points $P_i = \gamma(i)$, $i = 1, 2, 3, \cdots$, on the curve γ. This is an infinite sequence on the compact set M, and therefore[13] it has a limit point, P. Let N be a sufficiently small open neighborhood of P having the property that every curve tangent to ξ^μ passing through N can be distorted in N so that, while remaining timelike, it passes through P itself. Then for any positive number s, there must be a $t > s$ with $\gamma(t)$ in the neighborhood N (since P is a limit point of the P_i), and a $t' > s$ with $\gamma(t')$ not in N (since γ has no future endpoint). That is, γ must pass into and then out of the neighborhood N an infinite number of times. By distorting γ on two of its "passages" through N so that it intersects P, we obtain a closed timelike curve, beginning and ending at P.

Since this possibility has been excluded by hypothesis, we conclude that every such curve γ must have a future endpoint. This endpoint cannot occur in the interior of M, for ξ^μ vanishes nowhere, nor on S, since M is isochronous. Thus, the future endpoint of γ lies on S'. To summarize, through every point P of M, there exists one and only one curve γ everywhere tangent to ξ^μ, and this curve has one endpoint on S, and the other on S'.

Through each point P of M draw the curve γ. Let the curve γ be written in coordinate form, $x^\mu(t)$, where the parameter t is now defined by the equations

$$dx^\mu/dt = \xi^\mu, \quad x^\mu(t) \text{ lies on } S.$$

Suppose t assumes the values t_1 and t_2 at P and S', respectively. Now, define a scalar field, φ, on M by its value at each point P:

$$\varphi \equiv t_1/t_2.$$

The field φ now plays the role of the Morse function.[14] The surface S is given by $\varphi = 0$, and the surface S' by $\varphi = 1$. The one parameter family of surfaces, $\varphi = \text{const}$, pass through every point of M. Finally, the congruence ξ^μ provides a one-to-one correspondence between any two surfaces of this family, Hence, S and S' are diffeomorphic, and $M = S \times [0, 1]$.

An almost identical argument suffices to demonstrate the following theorem.

Theorem 3: Every compact geometry (without boundary) has a closed timelike curve.

Note added in proof: This theorem has been proved by R. W. Bass and L. Witten, Rev. Mod. Phys., **29**, 452 (1957), and by E. H. Kronheimer and R. Penrose, in a preprint, "On the Structure of Causal Spaces."

We now discuss two applications of these results. Suppose first that it has been established that at the present epoch the universe has a compact spacelike surface topologically a 3-sphere (i.e., the closed Friedmann model is a reasonable approximation). One could imagine that, during the highly contracted phase, the inhomogeneities have grown[15] to the extent that some topological "twisting" takes place. The universe than emerges from the contracted phase with expanding spacelike sections (the expanding phase of the Friedmann model). In this way, the universe might hope to "bounce" while avoiding a singularity. But according to our theorem, any such "topological twisting" can occur only if the universe is acausal.

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[10] See Ref. 9, p. 93.

[11] Here and elsewhere, S^n denotes the n-sphere, and an × denotes the topological product.

[12] See Ref. 7.

[13] See, for example, E. J. McShane and T. Botts, *Real Analysis* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1959), p. 51.

[14] J. Milnor, *Morse Theory* (Princeton University Press, Princeton, New Jersey, 1963).

[15] See E. Lifschitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963).

As a second application, consider the following proposal for the construction of a quantized gravitational theory.[16] One envisions a wave functional of positive definite metrics on a 3-manifold ("3-geometries"), evaluated by a (as yet to be spelled out in full detail) Feynman sum over "kinematically possible histories" (i.e., 4-geometries of the correct signature). That is,

$$\Psi({}^3\mathcal{G}) = \sum_{\text{4-geometries}} \exp\left[-\frac{i}{\hbar}\int R(-g)^{\frac{1}{2}}\, d^4x\right].$$

Wheeler[17] has expressed the hope that in this way there will emerge a "foam-like structure" for space, i.e., resonating states between 3-geometries of different topologies. Our two theorems state that one may always find a kinematically possible history which contributes to the Feynman sum for an arbitrary configuration of the 3-geometry, but that if any variation in the topology of that geometry (the foam-like structure) is to occur, acausal histories must be admitted into the Feynman sum. Thus, it is hard to see how the "foam structure" can be consistent with causality on the macroscopic level.

II. COVERING MANIFOLDS

The universal covering manifold has been a useful concept in the global analysis of relativistic world models.[18] Intuitively, a covering manifold is a second, larger, manifold, obtained from a given manifold, in which local properties are retained, but some of the topological features have been lost. The universal covering manifold is the "largest" covering manifold. Covering manifolds are defined in general without reference to other structures (i.e., a Lorentz metric) on the manifold in question. Our goal is to define a class of covering manifolds, these definitions utilizing the metric, in such a way that certain selected topological features are retained.

Let M be a 4-manifold. A covering manifold[19] of M is defined as a second 4-manifold, M', along with a map, φ, of M' onto M, such that for any point P of M, there exists an open neighborhood N of P such that $\varphi^{-1}(N)$ is a collection of open sets of M', each mapped diffeomorphically onto N by φ. Choose some fixed point P_0 of M, and let C be the set

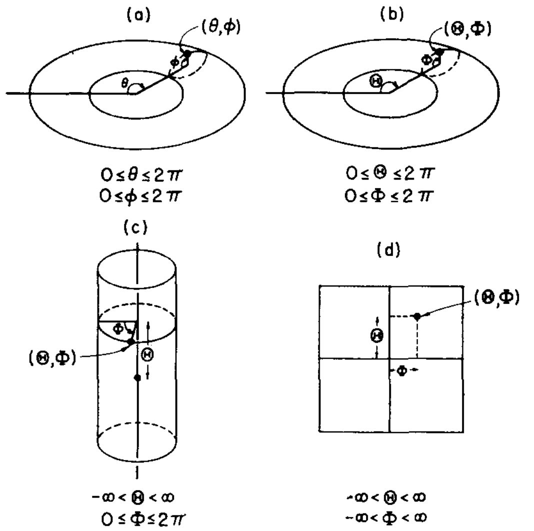

FIG. 1. An example of the torus (a), and three of its covering manifolds (b), (c), and (d). The covering manifolds are mapped onto the torus by the following equations: (b) $\boldsymbol{\theta} = \boldsymbol{\Theta}$, $\boldsymbol{\phi} = 2\boldsymbol{\Phi}$ (mod 2π); (c) $\boldsymbol{\theta} = 2\boldsymbol{\Theta}$ (mod 2π), $\boldsymbol{\phi} = \boldsymbol{\Phi}$; (d) $\boldsymbol{\theta} = \boldsymbol{\Theta}$ (mod 2π), $\boldsymbol{\phi} = \boldsymbol{\Phi}$ (mod 2π). Note that (d), the universal covering manifold, is simply connected.

$$C \equiv \{P, \gamma / P \text{ is a point of } M, \quad \gamma \text{ is a curve from } P \text{ to } P_0\}.$$

If (P, γ) and (P', γ') are elements of C, we write $(P, \gamma) \approx (P', \gamma')$ if $P = P'$, and γ and γ' are homotopic.[20] Then $\approx$ is an equivalence relation in C, and the equivalence classes define the points of the universal covering manifold[19] of M.

As an example of these two concepts, consider the 2-torus (Fig. 1). Points of the torus are labeled by the two angular coordinates θ and ϕ. Three examples of covering manifolds for the torus are shown in the figure, the last being the universal covering manifold.

From the definition, it is obvious that any tensor field residing on M determines uniquely a corresponding tensor field on any covering manifold M'. In particular, if M has a metric, then M' is assigned a unique metric.

Four elementary properties of the universal covering manifold should be mentioned.

(1) The universal covering manifolds resulting from two different choices of the initial point P_0 are identical (i.e., diffeomorphic).[19]

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[16] J. A. Wheeler, Ann. Phys. (N.Y.) **2**, 604 (1957); C. W. Misner, Rev. Mod. Phys. **29**, 497 (1957); H. Leutwyler, Phys. Rev. **134**, B1155 (1964).

[17] See Wheeler, Ref. 16.

[18] See Ref. 3, also S. Hawking, Phys. Rev. Letters **15**, 689 (1965).

[19] See, for example, J. G. Hocking and G. S. Young, *Topology* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961), p. 188.

[20] See, for example, A. H. Wallace, *Algebraic Topology* (Pergamon Press, Inc., New York, 1963), p. 63.

(2) The universal covering manifold is simply connected.[19]

(3) Every covering manifold of M is covered by the universal covering manifold of M.[19] (That is, the universal covering manifold is the "largest" covering manifold.)

(4) Let M be a geometry, and M' be its universal covering manifold. We obtain from the covering a unique metric on M'. Then M' is isochronous.[21]

Before discussing other covering manifolds, it is convenient to point out certain additional properties of the universal covering manifold in the presence of other structures on the manifold. We mention one property that the universal covering manifold does, and one property that it does not, have.

Let M be a geometry. Suppose in M we have a spacelike 3-surface S which is complete, i.e., S does not have any boundary points in M. In the universal covering manifold, M', of M, we may identify a corresponding surface S'.[22] Now, it is possible in general for some timelike curve in M to intersect S more than once, but in M' no timelike curve can intersect S' more than once. The proof is not difficult. Suppose that, for purposes of contradiction, there were in M' a timelike curve γ whose endpoints are the points P and P' of S'. We may assume without loss of generality that γ intersects S' only at P and P'. S' is two sided (i.e., a continuous choice of the forward light cone can be made on S'), since M' is isochronous. Consider a second curve $\bar{\gamma}$ which also begins at P, ends at P', and in addition coincides with γ in a finite neighborhood of P and P'. We construct $\bar{\gamma}$ as follows: $\bar{\gamma}$ begins at P, coincides with γ along a finite stretch, and thereafter remains within a neighborhood of S', staying always on the same side of S'. Then, at some point, $\bar{\gamma}$ crosses S', and again stays within a neighborhood of S', but now remaining on the other side. Finally, $\bar{\gamma}$ joins γ near P', and coincides with γ along the final stretch to P'. Now, $\bar{\gamma}$ crosses S' just once. Since M' is simply connected, γ can be continuously distorted into $\bar{\gamma}$, while always keeping fixed the finite stretches near P and P'. The impossibility of this follows from the fact that on distortion the number of intersections of γ with the complete surface S' must increase or decrease always by two, while $\bar{\gamma}$ intersects S' but once.

One might imagine that in general no closed timelike curve can occur on the universal covering manifold of a geometry M. (This is true in two dimensions.) That this is not the case in four dimensions is shown by an example. It is well known[21] that a continuous nonzero vector field can be placed on the 3-sphere, S^3. Hence, a signature $(-, +, +)$ metric can be placed thereon. Define a geometry M by

$$M \equiv S^3 \text{ (above metric)} \times (-\infty, +\infty) \text{ (usual metric)}.$$

M has a closed timelike curve (since by Theorem 3 of Sec. I, there is one on S^3). But M is simply connected, and hence is its own universal covering manifold. In particular, this universal covering manifold has a closed timelike curve.

We now try to isolate certain of the above properties of the universal covering manifold by means of suitably selected covering manifolds.

Let M be a given geometry. Choose any point P_0 of M, and define, as before,

$$C \equiv \{P, \gamma / P \text{ a point of } M, \quad \gamma \text{ is any curve from } P \text{ to } P_0\}.$$

Now, consider the equivalence relation: $(P, \gamma) \approx (P', \gamma')$ if $P = P'$, and if a timelike vector, when continuously transferred from P_0 to P along γ and back to P_0 along γ', does not reverse its time direction. The equivalence classes define a new covering manifold, which we call the Lorentz covering manifold of M. It is easy to verify that it has the following properties.

(1) The Lorentz covering manifold is isochronous.

(2) If M is isochronous, its Lorentz covering manifold is M itself.

(3) Any covering manifold of M which is isochronous covers the Lorentz covering manifold of M. (Thus, the Lorentz covering manifold is the "smallest" isochronous covering manifold.)

Next, suppose we have an isochronous geometry M, and that in M we have a complete spacelike surface S. Choose a point P_0 of M and define the set C as above. We now assign an integer, the index, to each element of C as follows. Each time γ crosses or meets S going forward in time, assign the integer $+1$, and each time going backward, assign a -1. Add the $+1$'s and -1's to obtain the index of the curve γ. Consider now the equivalence relation in C: $(P, \gamma) \approx (P', \gamma')$ if $P = P'$ and γ and γ' have the same index. The equivalence classes define the covering manifold of M relative to S. It has the properties:

(1) If S is a complete surface in the geometry M, and S' is a corresponding[22] surface in the relative covering manifold M', then no timelike curve in M' intersects S' more than once.

(2) Let M'' be another covering manifold of M having the property that no curve in M'' meets S'' exactly twice, once from each side of S''. Then M''

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[21] L. Markus, Ann. Math. **62**, 411 (1955).

[22] The image of S in M' may consist of several disjoint parts. Here and elsewhere, we choose any one connected part to serve as S'.

covers the relative covering manifold of M. (That is, the relative covering manifold is the "smallest" covering manifold embodying the other important property of the universal covering manifold.)

The relative covering manifold may also be characterized by the property that it is the smallest covering manifold which is divided into two disjoint parts by the surface S'.

The importance of covering manifolds in the study of singularities stems from the fact that a geometry M has a singularity[23] if and only if there is a singularity in every covering manifold of M. The technique,[24] then, is to apply the various theorems about singularities not to the geometry M, but rather to one of its covering manifolds. One is thus able to weaken the hypotheses of many of the singularity theorems.

More specifically, consider those theorems[25] which contain in their hypotheses, among other conditions, the requirements that the geometry M:

(1) be isochronous,
(2) has a complete spacelike surface S,
(3) has the property that no timelike curve intersects S more than once.

As we have seen, the universal covering manifold of a geometry M automatically satisfies conditions (1) and (3), even if M does not. There remains, therefore, the investigation of the conditions under which the other hypotheses [besides (1) and (3)] of each singularity theorem are passed from the geometry M to its universal covering manifold. This question has been investigated by Hawking[24] for the case in which S is not required to be compact.

But when[26] S is required to be compact (the "closed universe" theorems), a difficulty arises. This is the difficulty that the compactness of S is not in general passed to the universal covering manifold of M. However, if S is two-sided, its compactness is preserved under (a) the passage to the Lorentz covering manifold, and then (b) the passage to the relative covering manifold. In short, the two covering manifolds defined here can be used to weaken the hypotheses of even the theorems about singularities in closed universes. A case by case analysis of these theorems is not attempted here.

ACKNOWLEDGMENT

The author wishes to thank Professor J. A. Wheeler for many valuable discussions during the course of this work.

This research was supported in part by the United States Air Force Office of Research, Air Research and Development Command, under Contract No. AF 49-638-1545.

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[23] C. W. Misner, J. Math. Phys. **4**, 924 (1963).

[24] S. Hawking, Phys. Rev. Letters **15**, 689 (1965).

[25] R. Penrose, Phys. Rev. Letters **14**, 57 (1965), S. Hawking, Proc. Roy. Soc. (London) **A294**, 511 (1966); **A295**, 490 (1966).

[26] S. Hawking, Phys. Rev. Letters **17**, 444 (1966); R. Geroch, *ibid.* **17**, 445 (1966).